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Invariants of a System of Linear Partial Differential Equations, and the Theory of Congruences of Rays.

By E. J. Wilczynski.*

In a series of papers, published in the Transactions of the American Mathematical Society, the author has laid the foundations for a theory of ruled surfaces, based upon the consideration of the invariants of a system of ordinary linear differential equations. In the present paper, these considerations are extended to partial differential equations. The paper confines itself, for the most part, to a special case, and this for two reasons. In the first place, the complication of the more general cases is such as to make it appear unwise to attempt their treatment at present; moreover, the real essentials of the method appear just as well in a special case. In the second place, the special problem considered has its own intrinsic interest, as it leads to a theory of congruences. A whole geometry seems to be possible on this basis. The theory of curves, of surfaces, of complexes, etc., can all be built up, and have in part already been built up, by considerations which are of the same character as those discussed in this paper.

§1. *Determination of the most general transformation which converts a general system of linear partial differential equations of the first order into another of the same kind.*

Let us consider a system of q homogeneous linear partial differential equations of the first order

$$\sum_{k=1}^{n}\sum_{i=1}^{m} a_{ki}^{(\mu)} \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{n} b_k^{(\mu)} y_k = 0, \quad (\mu = 1, 2, \ldots . q) \tag{1}$$

with the n unknown functions $y_1, \ldots . y_n$ of the m independent variables

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$x_1, \dots x_m$. We wish to find the most general point transformation

$$y_k = g_k(\xi_1, \dots \xi_m;\ \eta_1, \dots \eta_n), \quad x_l = f_l(\xi_1, \dots \xi_m;\ \eta_1, \dots \eta_n),$$
$$(k = 1, 2, \dots n), \quad (l = 1, 2, \dots m), \tag{2}$$

which will transform (1) into another system of the same form.

We have from (2)

$$dy_k = \sum_{\lambda=1}^{m} \frac{\partial g_k}{\partial \xi_\lambda} d\xi_\lambda + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} d\eta_\mu,$$
$$dx_k = \sum_{\lambda=1}^{m} \frac{\partial f_k}{\partial \xi_\lambda} d\xi_\lambda + \sum_{\mu=1}^{n} \frac{\partial f_k}{\partial \eta_\mu} d\eta_\mu.$$

But after the transformation η_k is a function of $\xi_1, \dots \xi_m$, so that

$$d\eta_k = \sum_{\nu=1}^{m} \frac{\partial \eta_k}{\partial \xi_\nu} d\xi_\nu,$$

and therefore

$$dy_k = \sum_{\lambda=1}^{m} \left(\frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) d\xi_\lambda,$$
$$dx_k = \sum_{\lambda=1}^{m} \left(\frac{\partial f_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial f_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) d\xi_\lambda.$$

But we also have

$$dy_k = \sum_{\nu=1}^{m} \frac{\partial y_k}{\partial x_\nu} dx_\nu,$$

whence, substituting the above values for dy_k and dx_ν,

$$dy_k = \sum_{\lambda=1}^{m} \left(\frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) d\xi_\lambda = \sum_{\nu=1}^{m} \left[\frac{\partial y_k}{\partial x_\nu} \sum_{\lambda=1}^{m} \left(\frac{\partial f_\nu}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial f_\nu}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right) \right] d\xi_\lambda,$$
$$(k = 1, 2, \dots n).$$

These equations must be satisfied for all values of $d\xi_1, \dots d\xi_m$, so that we must have

$$\frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} = \sum_{\nu=1}^{m} \frac{\partial y_k}{\partial x_\nu} \left(\frac{\partial f_\nu}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial f_\nu}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda} \right), \tag{3}$$
$$(k = 1, 2, \dots n;\ \lambda = 1, 2, \dots m),$$

a system of mn equations whose solution would give the expressions for $\frac{\partial y_k}{\partial x_\nu}$ in terms of the quantities $\frac{\partial \eta_k}{\partial \xi_\nu}$. These expressions must be linear if the transformed of system (1) is to be again a linear system. But this will be so if, and only if, all of the partial derivatives $\frac{\partial f_\nu}{\partial \eta_\mu}$ vanish, i. e. if the f_ν quantities are functions of $\xi_1, \ldots. \xi_m$ alone. Equations (3) then become

$$\sum_{\nu=1}^{m} \frac{\partial f_\nu}{\partial \xi_\lambda} \frac{\partial y_k}{\partial x_\nu} = \frac{\partial g_k}{\partial \xi_\lambda} + \sum_{\mu=1}^{n} \frac{\partial g_k}{\partial \eta_\mu} \frac{\partial \eta_\mu}{\partial \xi_\lambda}, \tag{3a}$$

where now obviously the right members must be linear functions of $\eta_1, \ldots. \eta_n, \frac{\partial \eta_k}{\partial \xi_1}$, etc. But this is so only if

$$\frac{\partial g_k}{\partial \eta_\mu} = \alpha_{k\mu}$$

is a function of $\xi_1, \ldots. \xi_m$ alone, i. e. if

$$g_k = \alpha_{k0} + \alpha_{k1} y_1 + \ldots. + \alpha_{kn} y_n.$$

But the right member of (3a) must be linear and *homogeneous* in $\eta_1, \ldots. \eta_n$, $\frac{\partial \eta_1}{\partial \xi_1}$, etc. Therefore

$$\frac{\partial \alpha_{k0}}{\partial \xi_\lambda} = 0, \quad (\lambda = 1, 2, \ldots\ldots m;\ k = 1, 2, \ldots. n),$$

i. e. α_{k0} can only be a constant. But it must even be zero, for else the new system obtained from (1) by the transformation, while again linear, would not be homogeneous.

We have, therefore, the following result: *The most general transformation which converts system* (1) *into another of the same form, is homogeneous and linear in the dependent variables, but entirely arbitrary in the independent variables. The general form of the transformation is*

$$y_k = \alpha_{k1}\eta_1 + \alpha_{k2}\eta_2 + \ldots. + \alpha_{kn}\eta_n, \qquad (k = 1, 2, \ldots. n),$$
$$x_l = f_l(\xi_1, \xi_2, \ldots. \xi_m), \qquad (l = 1, 2, \ldots. m),$$

where α_{ki} and f_l are arbitrary functions of $\xi_1, \ldots. \xi_m$, subject of course to the restriction that the determinant

$$|\alpha_{ki}|$$

and the Jacobian of $f_1, \ldots. f_m$ with respect to $\xi_1, \ldots. \xi_m$ shall not vanish.

This theorem applies of course only to a *general* system of form (1) and to transformations which have no special relation to the system considered. The same remark applies to Staeckel's proof of the corresponding theorem for a single ordinary linear differential equation* as well as to my own proof of the corresponding theorem for a system of such equations.† It is easy enough to see that the theorem may be extended to cover systems of partial differential equations containing higher derivatives than the first, moreover not necessarily linear but still homogeneous. All of these results are simple consequences of Lie's theory.

§2.—*Introduction of the special problem. General remarks on the calculation of the invariants.*

We shall, in this paper, confine our attention exclusively to the special case of two equations with two dependent and two independent variables, viz.

$$\left.\begin{aligned}\Omega &= ay_1 + by_2 + cz_1 + dz_2 + ey + fz = 0,\\ \Omega' &= a'y_1 + b'y_2 + c'z_1 + d'z_2 + e'y + f'z = 0,\end{aligned}\right\} \tag{1}$$

where

$$y_k = \frac{\partial y}{\partial x_k}, \quad z_k = \frac{\partial z}{\partial x_k}, \quad (k = 1, 2) \tag{2}$$

and where $a, b, \ldots. f'$ are functions of x_1 and x_2.

Let $\alpha, \beta, \gamma, \delta, g$ and h be arbitrary functions of ξ_1 and ξ_2 such that the determinants $\alpha\delta - \beta\gamma$ and $\dfrac{\partial(g, h)}{\partial(\xi_1, \xi_2)}$ do not vanish. Then, as we have seen in §1, the most general transformation which converts (1) into a system of the same character is of the form

$$\left.\begin{aligned}y &= \alpha\eta + \beta\zeta, & z &= \gamma\eta + \delta\zeta;\\ x_1 &= g(\xi_1, \xi_2), & x_2 &= h(\xi_1, \xi_2).\end{aligned}\right\} \tag{3}$$

But clearly one may also consider instead of (1), any system of the form

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0 \tag{4}$$

for these two systems are entirely equivalent. Since the coefficients of (4) are simple combinations of those of (1), the operation of replacing (1) by (4) may also be looked upon as a transformation.

In calculating the invariants of (1) we shall apply these various transformations successively. We shall call those functions of the coefficients of (1) which remain invariant under the transformation

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* Staeckel, Crelle's Journal, Vol. 111.

† Wilczynski, Am. Jour. of Math., 1901.

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \tag{5}$$

where α, β, γ, δ are arbitrary functions of x_1 and x_2, *pseudo-invariants.* Those functions of pseudo-invariants, which are invariant under the additional transformation which corresponds to the substitution of (4) in place of (1), shall be called *seminvariants.* Finally, those functions of seminvariants, which do not change when the independent variables are arbitrarily transformed, shall be called *invariants.* A similar nomenclature is to be applied to the covariants.

§3.—*Determination of the pseudo-invariants.*

If we apply transformation (5) to system (1), a new system

$$\bar{\Omega} = 0, \quad \bar{\Omega}' = 0$$

is obtained, whose coefficients are

$$\left.\begin{aligned} \bar{a} &= a\alpha + c\gamma, \quad \bar{c} = a\beta + c\delta, \\ \bar{b} &= b\alpha + d\gamma, \quad \bar{d} = b\beta + d\delta, \\ e &= a\alpha_1 + b\alpha_2 + c\gamma_1 + d\gamma_2 + e\alpha + f\gamma, \\ \bar{f} &= a\beta_1 + b\beta_2 + c\delta_1 + d\delta_2 + e\beta + f\delta, \end{aligned}\right\} \tag{6}$$

while $\bar{a}'$, $\bar{b}'$, etc., are given by the same equations if every letter a, b, c, etc., be replaced by a', b', c', etc.

The transformations (5) form an infinite group, which may be defined by differential equations (in the sense of Lie). Therefore, Lie's theory of such groups may be applied. The most general infinitesimal transformation of this group will be obtained by putting

$$\alpha = 1 + \lambda\delta t, \quad \beta = \mu\delta t, \quad \gamma = \nu\delta t, \quad \delta = 1 + \rho\delta t,$$

where δt is an infinitesimal and where λ, μ, ν, ρ are arbitrary functions of x_1 and x_2. The corresponding infinitesimal variations of a, b, c, etc., may be easily computed from (6). They are

$$\left.\begin{aligned} \delta a &= (\lambda a + \nu c)\,\delta t, \quad \delta c = (\mu a + \rho c)\,\delta t, \\ \delta b &= (\lambda b + \nu d)\,\delta t, \quad \delta d = (\mu b + \rho d)\,\delta t, \\ \delta e &= (\lambda_1 a + \lambda_2 b + \nu_1 c + \nu_2 d + \lambda e + \nu f)\,\delta t, \\ \delta f &= (\mu_1 a + \mu_2 b + \rho_1 c + \rho_2 d + \mu e + \rho f)\,\delta t, \\ &\quad \text{etc.} \qquad\qquad \text{etc.} \end{aligned}\right\} \tag{7}$$

If any function F of these quantities is an absolute invariant under transformation (5), δF must vanish for arbitrary values of λ, μ, ν, ρ and of their various derivatives. The system of partial linear differential equations thus obtained for F is always a complete system according to Lie's theory. Although we shall be able to obtain most of our invariants without integrating such systems, we shall always use them to furnish the proof of the completeness of the system of invariants obtained otherwise.

From equations (6) it is obvious that

$$\left.\begin{array}{ll} i\ = ad - bc\,, & i' = a'd' - b'c',\\ k\ = ac' - a'c, & l\ = bd' - b'd,\\ m = ad' - b'c, & n = a'd - bc', \end{array}\right\} \qquad (8)$$

are relative pseudo-invariants, their quotients being absolute pseudo-invariants. Each of these functions satisfies the equation

$$\bar{F} = (\alpha\delta - \beta\gamma)\,F.$$

These pseudo-invariants involve merely the eight coefficients a, b, c, d; a', b', c', d'. If we set up the complete system of partial differential equations satisfied by pseudo-invariants of this kind, we find that it contains 4 independent equations and 8 variables. Therefore, there are only four independent absolute, and only five relative pseudo-invariants depending on these variables.

In fact, there is a syzygy between the quantities (8), namely,

$$ii' - mn - kl = 0, \qquad (9)$$

and we shall take

$$i,\ i',\ k,\ l \text{ and } t = m - n \qquad (10)$$

as our system of pseudo-invariants, for these five are independent. That this is so may be easily seen if we put

$$b' = 0, \quad c' = d' = 1.$$

For then

$$i = ad - bc, \quad i' = a', \quad k = a - a'c, \quad l = b, \quad t = a + b - a'd,$$

and the Jacobian of these five functions with respect to the five variables a, b, c, d, a', disregarding sign, is

$$a'\,(a - b + a'd),$$

and this does not vanish identically.

If we set up the system of partial differential equations, satisfied by the pseudo-invariants depending on the quantities

$$\left.\begin{array}{l} a\,,\ldots.\,d\,;\ a'\,,\ldots.\,d'\,;\ e\,,\ f\,; \\ a_k,\ldots.\,d_k;\ a'_k,\ldots.\,d'_k;\ e',\ f';\ (k=1,\ 2), \end{array}\right\} \tag{11}$$

we find that it consists of twelve independent equations with 28 independent variables. There are therefore 16 independent absolute or 17 relative pseudo-invariants containing these coefficients and their first derivatives.

In order to construct them we put

$$\left.\begin{array}{ll} p=e-a_1-b_2\,, & p'=e'-a'_1-b'_2\,, \\ q=f-c_1-d_2\,, & q'=f'-c'_1-d'_2\,. \end{array}\right\} \tag{12}$$

It is easy to see that the pairs of quantities

$$p,\ q;\ \ p',\ q';\ \ a,\ c;\ \ b,\ d;\ \ a',\ c';\ \ b',\ d';$$

are all transformed by the same equations, i. e. as we shall say are *cogredient.* Therefore the determinant of any two of these pairs is a pseudo-invariant.

Let us put

$$\left.\begin{array}{ll} r=aq-cp\,, & r'=a'q'-c'p'\,, \\ s=bq-dp\,, & s'=b'q'-d'p'\,. \end{array}\right\} \tag{13}$$

Then these quantities are pseudo-invariants.

Moreover if ϕ and ψ are two pseudo-invariants, for which

$$\bar{\phi}=\Delta^n\phi,\quad \bar{\psi}=\Delta^n\psi,\quad \Delta=\alpha\delta-\beta\gamma.$$

we have

$$\frac{\bar{\phi}_k}{\bar{\phi}}-\frac{\bar{\psi}_k}{\bar{\psi}}=\frac{\phi_k}{\phi}-\frac{\psi_k}{\psi}\,,\qquad (k=1,\ 2),$$

where

$$\phi_k=\frac{\partial\phi}{\partial x_k}\,,\quad \psi_k=\frac{\partial\psi}{\partial x_\kappa}\,.$$

Therefore,

$$\overline{\phi_k\psi-\phi\psi_k}=\Delta^{2n}\left(\phi_k\psi-\phi\psi_k\right). \tag{14}$$

Let us denote $\phi_k\psi-\phi\psi_k$ by $(\phi_k,\ \psi)$ and call it the *Wronskian* of ϕ and ψ with respect to x_k. It is clear then that the Wronskians of two such pseudo-invariants are again pseudo-invariants.

We can, therefore, write down the following list of 17 pseudo-invariants, all of which contain only the 28 quantities (11):

$$i,\ i',\ k,\ l,\ t;\ (i_k,\ t),\ (i'_k,\ t),\ (k_k,\ t),\ (l_k,\ t);\ r,\ s,\ r',\ s';\qquad (k=1,\ 2). \tag{15}$$

They are independent. For the first five are independent of each other, as we have already seen. Each of the next eight contains a quantity which is not contained in any preceding one, so that the first thirteen are independent. The last four contain the additional variables e, f, e', f' in four independent combinations, so that there can be no relation between the seventeen pseudo-invariants (15).

We continue our search for pseudo-invariants by asking for all those which contain, besides the variables (11), the second derivatives of $a \ldots\ldots d$, $a' \ldots\ldots d'$, and the first derivatives of e, f, e', f', i. e. altogether sixty quantities. The system of partial differential equations will consist, in this case, of twenty-four equations, obtained by equating to zero in δF the coefficients of λ, μ, ν, ρ and of the first and second derivatives of these functions with respect to x_1 and x_2. These twenty-four equations are independent. In order to abbreviate the proof of this statement, let us represent the matrix of the fourth order:

$$\begin{matrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{matrix}$$

by M, and a matrix of the fourth order all of whose terms are zero, by 0. If we now write down the matrix of the coefficients of our system of partial differential equations, we find in it the following determinant of the 24th order

$$\begin{vmatrix} 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & M \\ 0 & M & 0 & 2M_1 & M_2 & 0 \\ 0 & 0 & M & 0 & M_1 & 2M_2 \\ M & M_1 & M_2 & M_{11} & M_{12} & M_{22} \end{vmatrix}$$

where M_1 is the same matrix as M formed with a_1, b_1, etc. This determinant does not vanish. Therefore the 24 equations are independent.

This system of 24 equations must therefore have $60 - 24 = 36$ independent solutions, i. e. there must be 36 absolute, or 37 relative pseudo-invariants depending on these 60 variables.

We can easily form the twenty new pseudo-invariants. According to (14), the sixteen Wronskians

$$\left.\begin{array}{l} ((i_1, t)_1, t^2), \quad ((i'_1, t)_1, t^2), \quad ((k_1, t)_1, t^2), \quad ((l_1, t)_1, t^2), \\ ((i_2, t)_2, t^2), \text{ etc.,} \\ ((i_1, t)_2, t^2), \text{ etc.,} \\ ((i_2, t)_1, t^2), \text{ etc.,} \end{array}\right\} \tag{16}$$

are such pseudo-invariants. But only twelve of these are independent, for we have

$$((i_2, t)_1, t^2) = ((i_1, t)_2, t^2), \text{ etc.,} \tag{17}$$

as may be seen by direct computation.

The following eight quantities

$$\left.\begin{array}{llll} (r_1, t), & (r_2, t), & (r'_1, t), & (r'_2, t), \\ (s_1, t), & (s_2, t), & (s'_1, t), & (s'_2, t), \end{array}\right\} \tag{18}$$

are also pseudo-invariants. Together, we now have the twenty new pseudo-invariants. For, they are independent of the previous seventeen, and of each other, as each of them contains a variable not contained in any previous one, viz.

$$\begin{array}{llll} i_{11}, & i'_{11}, & k_{11}, & l_{11}, \\ i_{22}, & i'_{22}, & k_{22}, & l_{22}, \\ i_{12}, & i'_{12}, & k_{12}, & l_{12}, \\ af_1 - ce_1, & af_2 - ce_2, & a'f'_1 - c'e'_1, & a'f'_2 - c'e'_2, \\ bf_1 - de_1, & bf_2 - de_2, & b'f'_1 - d'e'_1, & b'f'_2 - d'e'_2, \end{array}$$

where the last eight are independent combinations of the eight variables $e_1, \ldots . f'_2$.

Of course this process for obtaining pseudo-invariants can be continued. There are 65 relative pseudo-invariants containing no higher derivatives of a, $b, \ldots . a'$ than the third, and no higher derivatives of $e, \ldots . f'$ than the second. For, the corresponding system of partial differential equations consists of 40 equations with 104 variables. These pseudo-invariants can all be obtained by continuing the Wronskian processes in an obvious manner. A simple induction completes the proof of the following theorem.

The pseudo-invariants of (1) *are all functions of* $i, i', k, l, t, r, s, r', s'$ *and of the quantities obtained by combining these with powers of* t *according to the Wronskian process.*

§ 4.—*Determination of the seminvariants.*

Let us now consider the system

$$\left.\begin{array}{l} \phi\Omega + \psi\Omega' = 0, \\ \chi\Omega + \omega\Omega' = 0, \end{array}\right\} \tag{19}$$

instead of (1), to which it is obviously equivalent if

$$\Delta = \phi\omega - \psi\chi \neq 0, \tag{20}$$

the functions ϕ, ψ, χ, ω of x_1 and x_2 being otherwise arbitrary. Then system (19) is of the same form as (1), and denoting its coefficients by capital letters, we have

$$\left.\begin{aligned} A &= \phi a + \psi a', & A' &= \chi a + \omega a', \\ B &= \phi b + \psi b', & B' &= \chi b + \omega b', \\ C &= \phi c + \psi c', & C' &= \chi c + \omega c', \\ D &= \phi d + \psi d', & D' &= \chi d + \omega d', \\ E &= \phi e + \psi e', & E' &= \chi e + \omega e', \\ F &= \phi f + \psi f', & F' &= \chi f + \omega f'. \end{aligned}\right\} \tag{21}$$

We can therefore regard the process of replacing system (1) by (19) as a transformation upon the coefficients of (1) by means of the equations (21), which are obviously again the equations of an infinite continuous group which can be defined by differential equations.

Seminvariants are such functions of the coefficients of system (1) and of their derivatives which are invariant under this transformation besides being pseudo-invariants.

Denoting systematically the various quantities formed for the new system by capital letters, we easily find

$$\left.\begin{aligned} I &= \phi^2 i + \phi\psi(m+n) + \psi^2 i', \\ I' &= \chi^2 i + \chi\omega(m+n) + \omega^2 i', \\ K &= \Delta k, \quad L = \Delta l, \quad T = \Delta t, \end{aligned}\right\} \tag{22}$$

so that k, l and t are at once seen to be seminvariants, as are also the combinations (k_i, t) and (l_i, t), and all others obtained by a repetition of this process.

We again introduce infinitesimal transformations by putting

$$\phi = 1 + \varepsilon\delta t, \quad \psi = \zeta\delta t, \quad \chi = \eta\delta t, \quad \omega = 1 + \theta\delta t,$$

where ε, ζ, η, θ are arbitrary functions and δt is an infinitesimal. From (22) we then find immediately

$$\left.\begin{aligned} \delta i &= 2\varepsilon i + \zeta(m+n), \\ \delta i' &= \eta(m+n) + 2\theta i', \\ \delta k &= (\varepsilon + \theta) k, \\ \delta l &= (\varepsilon + \theta) l, \\ \delta t &= (\varepsilon + \theta) t, \end{aligned}\right\} \tag{23}$$

where the infinitesimal factor δt has been omitted everywhere on the right members.

Remembering that

$$(i_k,\, t) = i_k t - i t_k, \quad ((m+n)_k,\, t) = (m_k + n_k)\, t - (m+n)\, t_k, \text{ etc.},$$

we find from (23), again omitting the infinitesimal factor,

$$\left.\begin{aligned} \delta(i_k,\, t) &= 3\varepsilon(i_k,\, t) + \zeta((m+n)_k,\, t) + \theta(i_k,\, t) + \zeta_k t(m+n) - \theta_k i t, \\ \delta(i'_k,\, t) &= \varepsilon(i'_k,\, t) + \eta((m+n)_k,\, t) + 3\theta(i'_k,\, t) + \eta_k t(m+n) - \varepsilon_k i' t, \\ \delta(k_k, t) &= 2\,(\varepsilon + \theta)(k_k,\, t), \qquad (k = 1,\, 2). \\ \delta(l_k,\, t) &= 2\,(\varepsilon + \theta)(l_k,\, t), \end{aligned}\right\} \tag{24}$$

From equations (12) and (13) we find

$$\left.\begin{aligned} \delta r &= 2\varepsilon r + \zeta u - \zeta_1 k - \varepsilon_2 i - \zeta_2 m, \\ \delta s &= 2\varepsilon s + \zeta v + \varepsilon_1 i + \zeta_1 n - \zeta_2 l, \\ \delta r' &= 2\theta r' + \eta u + \eta_1 k - \eta_2 n - \theta_2 i', \\ \delta s' &= 2\theta s' + \eta v + \eta_1 m + \theta_1 i' + \eta_2 l, \end{aligned}\right\} \tag{25}$$

where we have put

$$\left.\begin{aligned} u &= aq' - cp' + a'q - c'p, \\ v &= bq' - dp' + b'q - d'p, \end{aligned}\right\} \tag{26}$$

which quantities are manifestly pseudo-invariants which could be expressed algebraically in terms of the standard set.

Let us make a table of these infinitesimal transformations. At the head of each of the seventeen columns we place one of the functions

$$i,\, i',\, k,\, l,\, t; \quad (i_k,\, t),\, (i'_k,\, t),\, (k_k,\, t),\, (l_k,\, t); \quad r,\, s,\, r',\, s'.$$

To the left of each horizontal row we write one of the letters ε, ζ, η, θ, ε_1, etc. The infinitesimal transformation of any of the seventeen quantities i, i', etc., is then easily obtained by looking at the corresponding column in the table. Each of the quantities in that column is multiplied by the arbitrary function to the left of the row in which it is found, and the sum of all such products is the infinitesimal transformation required.

But this table, at the same time, serves another purpose. It gives the coefficients of the system of partial differential equations, whose solutions are the seminvariants depending upon the seventeen variables considered. For example, the quantities in the first row of the table are the coefficients of the equation

$$2i \frac{\partial F}{\partial i} + k \frac{\partial F}{\partial k} + l \frac{\partial F}{\partial l} + t \frac{\partial F}{\partial t} + 3\,(i_1,\, t)\, \frac{\partial F}{\partial (i_1,\, t)} + \text{etc.} = 0,$$

which is obtained by equating to zero the coefficients of ε in the general expression for δF. We may then speak of equation ε, equation ζ, etc.

The table now follows:

	i	i'	k	l	t	(i_1, t)	(i_2, t)	(i_1', t)	(i_2', t)	(k_1, t)	(k_2, t)	(l_1, t)	(l_2, t)	r	s	r'	s'
ε	$2i$	0	k	l	t	$3(i_1, t)$	$3(i_2, t)$	(i_1', t)	(i_2', t)	$2(k_1, t)$	$2(k_2, t)$	$2(l_1, t)$	$2(l_2, t)$	$2r$	$2s$	0	0
ζ	$m+n$	0	0	0	0	(m_1+n_1, t)	(m_2+n_2, t)	0	0	0	0	0	0	u	v	0	0
η	0	$m+n$	0	0	0	0	0	(m_1+n_1, t)	(m_2+n_2, t)	0	0	0	0	0	0	u	v
θ	0	$2i'$	k	l	t	(i_1, t)	(i_2, t)	$3(i_1', t)$	$3(i_2', t)$	$2(k_1, t)$	$2(k_2, t)$	$2(l_1, t)$	$2(l_2, t)$	0	0	$2r'$	$2s'$
ε_1	0	0	0	0	0	0	0	$-i't$	0	0	0	0	0	0	i	0	0
ζ_1	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	0	$-k$	n	0	0
η_1	0	0	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	0	k	m
θ_1	0	0	0	0	0	$-it$	0	0	0	0	0	0	0	0	0	0	i'
ε_2	0	0	0	0	0	0	0	0	$-i't$	0	0	0	0	$-i$	0	0	0
ζ_2	0	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	$-m$	$-l$	0	0
η_2	0	0	0	0	0	0	0	0	$t(m+n)$	0	0	0	0	0	0	$-n$	l
θ_2	0	0	0	0	0	0	$-it$	0	0	0	0	0	0	0	0	$-i'$	0

Let us consider the ten equations which are obtained by omitting equations ε and θ. None of these ten equations contains the partial derivatives of F with respect to any of the seven quantities k, l, t, (k_i, t), (l_i, t). These ten equations contain only the derivatives with respect to the other ten variables. But the determinant of these equations may be evaluated without much trouble. Disregarding its sign, it is equal to

$$t^4 (m+n)^2 i^2 i'^2 (t^2 - 4kl + 3ii')^2,$$

and therefore, in general, different from zero. But this shows that the partial derivatives of F with respect to the ten variables here considered must be zero. In other words, these seminvariants must be functions of k, l, t, (k_i, t), (l_i, t) alone. These latter quantities are themselves seminvariants. Therefore:

All seminvariants which contain no higher derivatives of the coefficients $a, b, \ldots\ldots d'$ *than the first and which contain no derivatives of the coefficients* e, f, e', f' *are functions of the following seven:*

$$k, l, t; \quad (k_i, t), (l_i, t); \qquad (i = 1, 2).$$

If we were to continue this method of determining the other seminvariants one step farther, our system of partial differential equations would show us, that there are two new seminvariants which cannot be formed from the previous seven by the Wronskian process. But, while it is possible to determine them, the amount of labor necessary for this is very great, and we prefer to avoid it by introducing from now on a normal form, which we shall consider as being preserved by all transformations of the differential equations (1).

This normal form can in general be obtained by merely solving a quadratic equation. We can make i and i' vanish by choosing ϕ, χ, ψ and ω appropriately. In fact if we put

$$\phi = m + n + \sqrt{(m+n)^2 - 4ii'}, \quad \psi = -2i,$$
$$\chi = m + n - \sqrt{(m+n)^2 - 4ii'}, \quad \omega = -2i,$$

the pseudo-invariants I and I' for the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

will be zero, as shown by equations (22). This can always be done unless

$$(m+n)^2 - 4ii' = t^2 - 4kl = 0,$$

which case we shall leave aside for the present.

Suppose that this reduction has been effected, so that $i = i' = 0$. The transformations (3) do not disturb these relations. But the transformations (21) do, in general. Equations (22), together with the condition $\phi\omega - \psi\chi \neq 0$, show that the only transformations of form (21), which leave invariant the system of equations $i = i' = 0$, are those for which either

$$\psi = \chi = 0 \quad \text{or} \quad \phi = \omega = 0.$$

But such transformations either convert every pseudo-invariant into itself, or change its sign, or at most interchange two of them, so that

$$k^2,\ l^2,\ t^2, \quad rr',\ r + r', \quad ss',\ s + s'$$

are invariant for all such transformations.

If, however, we disregard those transformations that interchange the two operators Ω and Ω' we may say that *for the normal form of our system of partial differential equations, which is defined by the conditions* $i = i' = 0$, *the quantities*

$$k,\ l,\ t,\ r,\ s,\ r',\ s'$$

and those obtained from them by the Wronskian process constitute a complete system of seminvariants.

This normal form can always be obtained by solving a quadratic, except if

$$t^2 - 4kl = 0.$$

§5.—*Determination of the invariants.*

If new independent variables be introduced by putting

$$\xi_k = \lambda_k(x_1, x_2), \qquad (k = 1, 2),$$

we find

$$\frac{\partial y}{\partial x_k} = \frac{\partial y}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_k} + \frac{\partial y}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_k}, \qquad (k = 1, 2).$$

If then we put

$$\frac{\partial \xi_1}{\partial x_1} = \lambda, \quad \frac{\partial \xi_1}{\partial x_2} = \mu, \quad \frac{\partial \xi_2}{\partial x_1} = \nu, \quad \frac{\partial \xi_2}{\partial x_2} = \rho,$$

we must assume $\lambda\rho - \mu\nu$ different from zero. If we denote the coefficients of the transformed system of differential equations again by $\bar{a}$, $\bar{b}$, etc., we find

$$\left.\begin{aligned}
\bar{a} &= a\lambda + b\mu, & \bar{a}' &= a'\lambda + b'\mu,\\
\bar{b} &= a\nu + b\rho, & \bar{b}' &= a'\nu + b'\rho,\\
\bar{c} &= c\lambda + d\mu, & \bar{c}' &= c'\lambda + d'\mu,\\
\bar{d} &= c\nu + d\rho, & \bar{d}' &= c'\nu + d'\rho,\\
\bar{e} &= e, & \bar{e}' &= e',\\
\bar{f} &= f, & \bar{f}' &= f'.
\end{aligned}\right\} \qquad (27)$$

From these equations it is easy to obtain the following:

$$\left.\begin{aligned} \bar{k} &= \lambda^2 k + \mu^2 l + \lambda\mu t, \\ \bar{l} &= \nu^2 k + \rho^2 l + \nu\rho t, \\ \bar{t} &= 2\lambda\nu k + (\lambda\rho + \mu\nu) t + 2\mu\rho l, \end{aligned}\right\} \tag{28}$$

whence

$$t^2 - 4kl \tag{29}$$

is seen to be an invariant.

It is necessary to obtain formulae which shall show how to find the effect of this transformation upon the derivatives of a given function. Let f be such a function of a, b, c, etc., and let $\bar{f}$ denote the corresponding function of $\bar{a}$, $\bar{b}$, etc., these being functions of ξ_1 and ξ_2, and therefore also of x_1 and x_2. We have

$$d\xi_1 = \lambda dx_1 + \mu dx_2, \quad d\xi_2 = \nu dx^1 + \rho dx_2$$

whence

$$\Delta dx_1 = \rho d\xi_1 - \mu d\xi_2, \quad \Delta dx_2 = -\nu d\xi_1 + \lambda d\xi_2,$$

where

$$\Delta = \lambda\rho - \mu\nu .$$

Therefore, we find

$$\Delta \frac{\partial \bar{f}}{\partial \xi_1} = \frac{\partial \bar{f}}{\partial x_1}\rho - \frac{\partial \bar{f}}{\partial x_2}\nu, \quad \Delta \frac{\partial \bar{f}}{\partial \xi_2} = -\frac{\partial \bar{f}}{\partial x_1}\mu + \frac{\partial \bar{f}}{\partial x_2}\lambda. \tag{30}$$

In particular, let the transformation be infinitesimal, so that

$$\lambda = 1 + \varkappa\varepsilon, \quad \mu = \pi\varepsilon, \quad \nu = \sigma\varepsilon, \quad \rho = 1 + \tau\varepsilon, \tag{31}$$

where ε is an infinitesimal, while $\varkappa$, π, σ and τ are arbitrary functions of x_1 and x_2. Then, neglecting infinitesimals of higher order,

$$\bar{f} = f + \delta f, \quad \Delta = 1 + (\varkappa + \tau)\,\varepsilon,$$

and, therefore, substituting in (30),

$$\left.\begin{aligned} \delta\left(\frac{\partial f}{\partial x_1}\right) &= \frac{\partial(\delta f)}{\partial x_1} - \left(\varkappa\frac{\partial f}{\partial x_1} + \sigma\frac{\partial f}{\partial x_2}\right)\varepsilon, \\ \delta\left(\frac{\partial f}{\partial x_2}\right) &= \frac{\partial(\delta f)}{\partial x_2} - \left(\pi\frac{\partial f}{\partial x_1} + \tau\frac{\partial f}{\partial x_2}\right)\varepsilon. \end{aligned}\right\} \tag{32}$$

We proceed to find the infinitesimal transformations of the various seminvariants. We have first (omitting the infinitesimal factor ε),

$$\left.\begin{aligned} \delta a &= \varkappa a + \pi b, & \delta a' &= \varkappa a' + \pi b', \\ \delta b &= \sigma a + \tau b, & \delta b' &= \sigma a' + \tau b', \\ \delta c &= \varkappa c + \pi d, & \delta c' &= \varkappa c' + \pi d', \\ \delta d &= \sigma c + \tau d, & \delta d' &= \sigma c' + \tau d', \\ \delta e &= \delta f = 0\ , & \delta e' &= \delta f' = 0\ . \end{aligned}\right\} \tag{33}$$

Substituting (31) in (28), we find

$$\left.\begin{aligned} \delta k &= 2\varkappa k + \pi t, \\ \delta l &= 2\tau l + \sigma t, \\ \delta t &= 2\sigma k + (\varkappa + \tau)\, t + 2\pi l. \end{aligned}\right\} \tag{34}$$

Therefore, making use of (32),

$$\left.\begin{aligned} \delta(k_1, t) &= 2\varkappa(k_1, t) + 2\pi(k_1, l) - \sigma(k_2, t) + \tau(k_1, t) \\ &\qquad + \varkappa_1 kt + \pi_1(t^2 - 2kl) - 2\sigma_1 k^2 - \tau_1 kt, \\ \delta(k_2, t) &= 3\varkappa(k_2, t) + \pi\,[2(k_2, l) - (k_1, t)] \\ &\qquad + \varkappa_2 kt + \pi_2(t^2 - 2kl) - 2\sigma_2 k^2 - \tau_2 kt, \\ \delta(l_1, t) &= -\sigma\,[2(k_1, l) + (l_2, t)] + 3\tau(l_1, t) \\ &\qquad + \tau_1 lt + \sigma_1(t^2 - 2kl) - 2\pi_1 l^2 - \varkappa_1 lt, \\ \delta(l_2, t) &= \varkappa(l_2, t) - \pi(l_1, t) - 2\sigma(k_2, l) + 2\tau(l_2, t) \\ &\qquad + \tau_2 lt + \sigma_2(t^2 - 2kl) - 2\pi_2 l^2 - \varkappa_2 lt. \end{aligned}\right\} \tag{35}$$

Similarly, remembering the defining equations (12) of p, q, p', q', we find

$$\left.\begin{aligned} \delta p &= -\varkappa_1 a - \pi_1 b - \sigma_2 a - \tau_2 b, & \delta p' &= -\varkappa_1 a' - \pi_1 b' - \sigma_2 a' - \tau_2 b', \\ \delta q &= -\varkappa_1 c - \pi_1 d - \sigma_2 c - \tau_2 d, & \delta q' &= -\varkappa_1 c' - \pi_1 d' - \sigma_2 c' - \tau_2 d', \end{aligned}\right\} \tag{36}$$

whence

$$\left.\begin{aligned} \delta r &= \varkappa r + \pi s - (\pi_1 + \tau_2)\, i, & \delta r' &= \varkappa r' + \pi s' - (\pi_1 + \tau_2)\, i', \\ \delta s &= \sigma r + \tau s + (\varkappa_1 + \sigma_2)\, i, & \delta s' &= \sigma r' + \tau s' + (\varkappa_1 + \sigma_2)\, i'. \end{aligned}\right\} \tag{37}$$

For the normal form for which $i = i' = 0$, we have

$$\left.\begin{aligned} \delta r &= \varkappa r + \pi s, & \delta s &= \sigma r + \tau s, \\ \delta r' &= \varkappa r' + \pi s', & \delta s' &= \sigma r' + \tau s'. \end{aligned}\right\} \tag{38}$$

Any function of the seminvariants k, l, t; (k_i, t), (l_i, t); r, s, r' and s', which is an absolute invariant under the most general transformation leaving the normal form $i = i' = 0$ unchanged, must therefore be a solution of the system of twelve partial differential equations which is represented by the following table. The letters $\varkappa$, τ, etc., to the left of each row indicate that the quantities in that row are the coefficients of the equation obtained by equating to zero the coefficient of $\varkappa$, τ, etc. in the general expression for δf. The letter at the top of each column indicates that the quantities in that column are the coefficients of the partial derivatives with respect to that quantity.

(39)

	k	l	t	(k^1, t)	(k_2, t)	(l_1, t)	(l_2, t)	r	r'	s	s'
$\varkappa$	$2k$	0	t	$2(k_1, t)$	$3(k_2, t)$	0	(l_2, t)	r	r'	0	0
π	t	0	$2l$	$2(k_1, l)$	$2(k_2, l) - (k_1, t)$	0	$-(l_1, t)$	s	s'	0	0
σ	0	t	$2k$	$-(k_2, t)$	0	$-2(k_1, l) - (l_2, t)$	$-2(k_2, l)$	0	0	r	r'
τ	0	$2l$	t	$+(k_1, t)$	0	$3(l_1, t)$	$2(l_2, t)$	0	0	s	s'
$\varkappa_1$	0	0	0	kt	0	$-lt$	0	0	0	0	0
π_1	0	0	0	$t^2 - 2kl$	0	$-2l^2$	0	0	0	0	0
σ_1	0	0	0	$-2k^2$	0	$t^2 - 2kl$	0	0	0	0	0
τ_1	0	0	0	$-kt$	0	lt	0	0	0	0	0
$\varkappa_2$	0	0	0	0	kt	0	$-lt$	0	0	0	0
π_2	0	0	0	0	$t^2 - 2kl$	0	$-2l^2$	0	0	0	0
σ_2	0	0	0	0	$-2k^2$	0	$t^2 - 2kl$	0	0	0	0
τ_2	0	0	0	0	$-kt$	0	lt	0	0	0	0

The determinant of equations π_2 and σ_2 is $t^2(t^2-4kl)$, and therefore not zero. This shows that (k_2, t) and (l_2, t) cannot occur in such an invariant. From equations π_1 and σ_1 one concludes that the invariants are also independent of (k_1, t) and (l_1, t). The system now reduces to one of four equations in seven variables, so that there must be three absolute or four relative invariants. Of these we already know one, viz.

$$\mathfrak{F} = t^2 - 4kl. \tag{40}$$

The other three are

$$\mathfrak{G} = lr^2 - trs + ks^2, \quad \mathfrak{G}' = lr'^2 - tr's' + ks'^2, \quad \mathfrak{H} = rs' - r's. \tag{41}$$

§6.—*General properties of invariants and covariants.*

We have noticed in the preceding paragraphs that every invariant is multiplied by a power of each of the determinants $\alpha\delta-\beta\gamma$, $\phi\omega-\psi\chi$, $\frac{\partial\xi_1}{\partial x_1}\frac{\partial\xi_2}{\partial x_2}-\frac{\partial\xi_1}{\partial x_2}\frac{\partial\xi_2}{\partial x_1}$, when all of the transformations here considered are performed. This could have been proved a priori without first computing the invariants, and is also true of the covariants. The proof for this would be very much like the corresponding proof in the theory of algebraic invariants.

If we make the special transformation

$$\bar{y} = \alpha y, \quad \bar{z} = \alpha z, \quad \alpha = \text{const.},$$

it is seen that every covariant must be a homogeneous function of y, z, y_k, z_k, etc., as well as of the coefficients a, b, etc., and of their derivatives.

Let ψ be any pseudo-covariant homogeneous of degree k in the coefficients, and of degree i in the variables. If we make the transformation

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z} \tag{1}$$

or

$$\bar{y} = \frac{\delta y - \beta z}{\alpha\delta-\beta\gamma}, \quad \bar{z} = \frac{-\gamma y + \alpha z}{\alpha\delta-\beta\gamma},$$

we find from equations (6)

$$\bar{a} = a\alpha + c\gamma, \text{ etc.},$$

and since we know that $\bar{\psi}$ will be equal to ψ multiplied by a power of $\alpha\delta-\beta\gamma$, the exponent of this power must be $\frac{k-i}{2}$ since $\bar{\psi}$ will be a homogeneous function of $\alpha, \beta, \gamma, \delta$ of degree $k-i$ while $\alpha\delta-\beta\gamma$ is homogeneous of degree 2.

For our purposes of calculation it was convenient to write our transformation in the form (1). But if we make it a rule to always write our equations of transformation solved with respect to the new variables, we should have

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z,$$

and correspondingly

$$\bar{\psi} = (\alpha\delta - \beta\gamma)^{\frac{i-k}{2}} \psi.$$

If ψ is also a semicovariant, we find that for the transformations, which correspond to replacing the system $\Omega = \Omega' = 0$ by

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

ψ is multiplied by

$$(\phi\omega - \psi\chi)^{\frac{k}{2}}.$$

Make now a particular transformation of the independent variables

$$\xi_1 = cx_1, \quad \xi_2 = cx_2,$$

where c is a constant. Then

$$\begin{aligned} &\bar{y} = y, \qquad \bar{y}_k = c^{-1} y_k, \quad \bar{y}_{kl} = c^{-2} y_{kl}, \text{ etc.,} \\ &\bar{a} = ca, \text{ etc.,} \quad \bar{e} = e, \text{ etc.} \end{aligned}$$

Every quantity is multiplied by some power of c. Let us say that a quantity multiplied by c^w is of weight w. Then clearly every invariant or covariant must be isobaric, and an arbitrary transformation of the dependent variables multiplies it by $(\lambda\rho - \mu\nu)^{\frac{w}{2}}$.

Let us unite our results in a theorem.

Every rational function of y, z, a, b, c, etc., which has the property of being absolutely invariant is a quotient of two homogeneous isobaric rational integral functions of these quantities which are relative invariants.

Let $I_{i,k}^{(w)}$ be such an integral rational invariant function, of weight w, of degree i in the variables, and of degree k in the coefficients. If the system

$$\Omega = \Omega' = 0$$

be transformed by putting first

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z, \quad \xi_1 = f_1(x_1, x_2), \quad \xi_2 = f_2(x_1, x_2),$$

and then by replacing it by

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

the transformed invariant is

$$\overline{I}_{ik}^{(w)} = (\alpha\delta - \beta\gamma)^{\frac{i-k}{2}} (\phi\omega - \psi\chi)^{\frac{k}{2}} (\lambda\rho - \mu\nu)^{\frac{w}{2}} I_{ik}^{(w)}.$$

Since the exponents must be integers if I is a rational invariant, we see further, that *both of the degrees and the weight of any invariant function must be even numbers.*

For invariants proper $i = 0$. From two such invariants of degrees k_1, k_2 of weights w_1, w_2 we can obtain an absolute invariant, if and only if two integers λ_1 and λ_2 can be determined so that

$$\lambda_1 k_1 + \lambda_2 k_2 = 0,$$
$$\lambda_1 w_1 + \lambda_2 w_2 = 0,$$

i. e. if, and only if,

$$k_1 w_2 - k_2 w_1 = 0.$$

For our four invariants we have the following table of weight and degrees:

Invariant.	Weight.	Degree.
$\mathfrak{F}$	4	4
$\mathfrak{H}$	2	4
$\mathfrak{G}$	4	6
$\mathfrak{G}'$	4	6

Therefore only one absolute invariant is obtained namely $\mathfrak{G}/\mathfrak{G}'$.

§7.—*Geometrical interpretation.*

Let us write the equations (1), putting into evidence the dependent variables, as follows:

$$\Omega(y, z) = 0, \quad \Omega'(y, z) = 0, \tag{42}$$

and let $y^{(k)}$, $z^{(k)}$ for $k = 1, 2, 3, 4$ be four pairs of simultaneous solutions, so that

$$\Omega(y^{(k)}, z^{(k)}) = 0, \quad \Omega'(y^{(k)}, z^{(k)}) = 0, \qquad (k = 1, 2, 3, 4). \tag{43}$$

Let us interpret $y^{(k)}$ and $z^{(k)}$ as the homogeneous coordinates of two points P_y and P_z in space. We shall have

$$y^{(k)} = f^{(k)}(x_1, x_2) \quad z^{(k)} = g^{(k)}(x_1, x_2) \qquad (k = 1, 2, 3, 4) \tag{44}$$

when these solutions are known. As x_1 and x_2 assume all of the values of which they are capable, P_y and P_z will describe two surfaces S_y and S_z, which moreover will be placed in a point-to-point correspondence, those points being corresponding points which belong to the same set of values (x_1, x_2).

An arbitrary change of the independent variables, clearly does not change these surfaces, nor does it change the point-to-point correspondence between them. The most general transformation of the form

$$y = \alpha(x_1, x_2)\eta + \beta(x_1, x_2)\zeta, \quad z = \gamma(x_1, x_2)\eta + \delta(x_1, x_2)\zeta,$$

clearly converts the points P_y and P_z into two others, P_η and P_ζ, situated on the line L_{yz} joining P_y and P_z. If then we consider the congruence of lines joining the corresponding points of the two surfaces S_y and S_z, this congruence will be the same for all of the systems of differential equations equivalent to (1) by transformations of the infinite group considered in this paper.

But it is necessary to examine this somewhat more closely, so that we may see to what extent this geometrical image is characteristic of our system of differential equations.

Let the functions $f^{(k)}$ and $g^{(k)}$ in (44) be any functions of x_1 and x_2 whatever, and put

$$\left.\begin{aligned} y &= c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)} + c_4 y^{(4)}, \\ z &= c_1 z^{(1)} + c_2 z^{(2)} + c_3 z^{(3)} + c_4 z^{(4)}, \end{aligned}\right\} \tag{45}$$

where $c_1 \ldots . c_4$ are arbitrary constants. By differentiation we find the following four equations:

$$\left.\begin{aligned} \frac{\partial y}{\partial x_1} &= \sum_{k=1}^{4} c_k \frac{\partial y^{(k)}}{\partial x_1}, & \frac{\partial y}{\partial x_2} &= \sum_{k=1}^{4} c_k \frac{\partial y^{(k)}}{\partial x_2}, \\ \frac{\partial z}{\partial x_1} &= \sum_{k=1}^{4} c_k \frac{\partial z^{(k)}}{\partial x_1}, & \frac{\partial z}{\partial x_2} &= \sum_{k=1}^{4} c_k \frac{\partial z^{(k)}}{\partial x_2}. \end{aligned}\right\} \tag{46}$$

We can eliminate $c_1, \ldots . c_4$ in two different ways, and thus obtain the two equations:

$$D_1 = \begin{vmatrix} y, & y^{(1)}, & \ldots . & y^{(4)} \\ \frac{\partial y}{\partial x_1}, & \frac{\partial y^{(1)}}{\partial x_1}, & \ldots . & \frac{\partial y^{(4)}}{\partial x_1} \\ \frac{\partial y}{\partial x_2}, & \frac{\partial y^{(1)}}{\partial x_2}, & \ldots . & \frac{\partial y^{(4)}}{\partial x_2} \\ \frac{\partial z}{\partial x_1}, & \frac{\partial z^{(1)}}{\partial x_1}, & \ldots . & \frac{\partial z^{(4)}}{\partial x_1} \\ \frac{\partial z}{\partial x_2}, & \frac{\partial z^{(1)}}{\partial x_2}, & \ldots . & \frac{\partial z^{(4)}}{\partial x_2} \end{vmatrix} = 0, \quad D_2 = \begin{vmatrix} z, & z^{(1)}, & \ldots . & z^{(4)} \\ \frac{\partial y}{\partial x_1}, & \frac{\partial y^{(1)}}{\partial x_1}, & \ldots . & \frac{\partial y^{(4)}}{\partial x_1} \\ \frac{\partial y}{\partial x_2}, & \frac{\partial y^{(1)}}{\partial x_2}, & \ldots . & \frac{\partial y^{(4)}}{\partial x_2} \\ \frac{\partial z}{\partial x_1}, & \frac{\partial z^{(1)}}{\partial x_1}, & \ldots . & \frac{\partial z^{(4)}}{\partial x_1} \\ \frac{\partial z}{\partial x_2}, & \frac{\partial z^{(1)}}{\partial x_2}, & \ldots . & \frac{\partial z^{(4)}}{\partial x_2} \end{vmatrix} = 0, \tag{47}$$

which are of the form of system (1) and are satisfied by the four pairs of functions $y^{(k)}$, $z^{(k)}$. Moreover, this system is essentially unique. We have eliminated in such a manner as to make $f = 0$, $e' = 0$. Had we eliminated differently, the reduction to this form could be accomplished *a posteriori*, and, moreover, as equations (21) show, in essentially one way.

In order that (47) may be a determinate system, it is, of course, necessary that each of the matrices

$$\left.\begin{Vmatrix} y^{(k)}, & \dfrac{\partial y^{(k)}}{\partial x_1}, & \dfrac{\partial y^{(k)}}{\partial x_2}, & \dfrac{\partial z^{(k)}}{\partial x_1}, & \dfrac{\partial z^{(k)}}{\partial x_2} \\ z^{(k)}, & \dfrac{\partial y^{(k)}}{\partial x_1}, & \dfrac{\partial y^{(k)}}{\partial x_2}, & \dfrac{\partial z^{(k)}}{\partial x_1}, & \dfrac{\partial z^{(k)}}{\partial x_2} \end{Vmatrix}\right\}, \quad (k = 1, 2, 3, 4) \qquad (48)$$

should contain at least one non-vanishing determinant of the fourth order. This we shall assume to be the case, and we will not here enter upon a discussion of the meaning of the exceptional cases.

We see that to every congruence there belongs, in general, a system of partial differential equations of the form (1), namely, the system (47).

But there are an infinite number of congruences giving rise to the same system of differential equations. In the first place it is clear that (for arbitrary constant coefficients c_k)

$$y = \sum_{k=1}^{4} c_k y^{(k)}, \quad z = \sum_{k=1}^{4} c_k z^{(k)}$$

also constitute a simultaneous system of solutions. But this means that every congruence, obtained from the given one by projective transformation, gives rise to the same system of differential equations. But more than that. Equations (47) merely express the fact that, if we denote by Y and Z any possible pair of solutions, while $y^{(k)}$, $z^{(k)}$ represent four particular pairs of solutions, the following conditions must be satisfied

$$\left.\begin{aligned} Y &= \sum_{k=1}^{4} \phi_k y^{(k)}, & Z &= \sum_{k=1}^{4} \phi_k z^{(k)}, \\ \frac{\partial Y}{\partial x_\lambda} &= \sum_{k=1}^{4} \phi_k \frac{\partial y^{(k)}}{\partial x_\lambda}, & \frac{\partial Z}{\partial x_\lambda} &= \sum_{k=1}^{4} \phi_k \frac{\partial z^{(k)}}{\partial x_\lambda}, \quad (\lambda = 1, 2). \end{aligned}\right\} \qquad (49)$$

If the quantities ϕ_k are constants we get the case just noticed. But the equations

(49) may be satisfied even if the quantities ϕ_k are not constants. All that is necessary for this, is that they satisfy the conditions

$$\sum_{k=1}^{4} y^{(k)} \frac{\partial \phi_k}{\partial x_\lambda} = 0, \quad \sum_{k=1}^{4} z^{(k)} \frac{\partial \phi_k}{\partial x_\lambda} = 0, \qquad (\lambda = 1, 2). \tag{50}$$

But this is equivalent to demanding that

$$\frac{\partial \phi_k}{\partial x_1} \text{ and } \frac{\partial \phi_k}{\partial x_2}$$

shall be the coordinates of two planes passing through L_{yz}.

If we denote the coordinates of any two such planes by $u^{(k)}$ and $v^{(k)}$, we may put

$$\frac{\partial \phi_k}{\partial x_1} = \lambda_1 u^{(k)} + \mu_1 v^{(k)}, \quad \frac{\partial \phi_k}{\partial x_2} = \lambda_2 u^{(k)} + \mu_2 v^{(k)}, \tag{51}$$

where λ_1, μ_1, λ_2, μ_2 are arbitrary functions subject merely to the integrability conditions

$$\frac{\partial}{\partial x_2}(\lambda_1 u^{(k)} + \mu_1 v^{(k)}) = \frac{\partial}{\partial x_1}(\lambda_2 u^{(k)} + \mu_2 v^{(k)}). \qquad (k = 1, 2, 3, 4). \tag{52}$$

We can express this result in another way. If $\lambda^{(1)} \ldots\ldots \lambda^{(4)}$, $\mu^{(1)} \ldots\ldots \mu^{(4)}$ are eight arbitrary functions, and we denote the minors of $t^{(1)} \ldots\ldots t^{(4)}$ in the two determinants

$$\begin{vmatrix} t^{(1)} & t^{(2)} & t^{(3)} & t^{(4)} \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ \lambda^{(1)} & \lambda^{(2)} & \lambda^{(3)} & \lambda^{(4)} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} t^{(1)} & t^{(2)} & t^{(3)} & t^{(4)} \\ y^{(1)} & \ldots & \ldots & y^{(4)} \\ z^{(1)} & \ldots & \ldots & z^{(4)} \\ \mu^{(1)} & \ldots & \ldots & \mu^{(4)} \end{vmatrix}$$

by $u^{(1)} \ldots\ldots u^{(4)}$ and $v^{(1)} \ldots\ldots v^{(4)}$ respectively, then $u^{(k)}$ and $v^{(k)}$ are the coordinates of any two planes passing through the line L_{yz} of the congruence. Determine the functions $\lambda^{(1)} \ldots\ldots \mu^{(4)}$ in the most general way, so as to satisfy the conditions

$$\frac{\partial u^{(k)}}{\partial x_2} - \frac{\partial v^{(k)}}{\partial x_1} = 0,$$

so that we may put

$$u^{(k)} = \frac{\partial \phi_k}{\partial x_1}, \quad v^{(k)} = \frac{\partial \phi_k}{\partial x_2},$$

Then the most general solution of system (1) is

$$Y=\sum_{k=1}^{4}\phi_k y^{(k)},\qquad Z=\sum_{k=1}^{4}\phi_k z^{(k)},$$

the four pairs of particular solutions $(y^{(k)}, z^{(k)})$ being such that not all of the determinants of the fourth order in the matrices (48) become equal to zero.

Under these assumptions we can deduce a system of partial differential equations for $u^{(k)}$ and $v^{(k)}$ similar in form to (1), and which may be spoken of as being adjoined to (1). For, with our interpretation, it represents the same congruence as (1), the lines of the congruence, however, being obtained as intersections of corresponding planes instead of as lines joining corresponding points.

As a matter of fact we shall have

$$\left.\begin{array}{ll}\sum u^{(k)}y^{(k)}=0, & \sum u^{(k)}z^{(k)}=0\\ \sum\left(u^{(k)}\frac{\partial y^{(k)}}{\partial x_1}+\frac{\partial u^{(k)}}{\partial x_1}y^{(k)}\right)=0, & \ldots\ldots\ldots\\ \sum\left(u^{(k)}\frac{\partial y^{(k)}}{\partial x_2}+\frac{\partial u^{(k)}}{\partial x_2}y^{(k)}\right)=0, & \ldots\ldots\ldots\\ \quad\text{etc.} & \end{array}\right\}\tag{53}$$

Now we have $\qquad\Omega(y^{(k)}, z^{(k)})=0,\quad \Omega'(y^{(k)}, z^{(k)})=0$

and therefore $\qquad\sum u^{(k)}\Omega(y^{(k)}, z^{(k)})=0,\quad$ etc.

whence, making use of (53),

$$a\sum u^{(k)}\frac{\partial y^{(k)}}{\partial x_1}+b\sum u^{(k)}\frac{\partial y^{(k)}}{\partial x_2}+c\sum u^{(k)}\frac{\partial z^{(k)}}{\partial x_1}+d\sum u^{(k)}\frac{\partial z^{(k)}}{\partial x_2}=0.$$

or again according to (53),

$$a\sum y^{(k)}\frac{\partial u^{(k)}}{\partial x_1}+b\sum y^{(k)}\frac{\partial u^{(k)}}{\partial x_2}+c\sum z^{(k)}\frac{\partial u^{(k)}}{\partial x_1}+d\sum z^{(k)}\frac{\partial u^{(k)}}{\partial x_2}=0.$$

We find thus the first of the following four equations:

$$\begin{aligned}&\sum y^{(k)}\left(a\frac{\partial u^{(k)}}{\partial x_1}+b\frac{\partial u^{(k)}}{\partial x_2}\right)+\sum z^{(k)}\left(c\frac{\partial u^{(k)}}{\partial x_1}+d\frac{\partial u^{(k)}}{\partial x_2}\right)=0,\\ &\sum y^{(k)}\left(a\frac{\partial v^{(k)}}{\partial x_1}+b\frac{\partial v^{(k)}}{\partial x_2}\right)+\sum z^{(k)}\left(c\frac{\partial v^{(k)}}{\partial x_1}+d\frac{\partial v^{(k)}}{\partial x_2}\right)=0,\\ &\sum y^{(k)}\left(a'\frac{\partial u^{(k)}}{\partial x_1}+b'\frac{\partial u^{(k)}}{\partial x_2}\right)+\sum z^{(k)}\left(c'\frac{\partial u^{(k)}}{\partial x_1}+d'\frac{\partial u^{(k)}}{\partial x_2}\right)=0,\\ &\sum y^{(k)}\left(a'\frac{\partial v^{(k)}}{\partial x_1}+b'\frac{\partial v^{(k)}}{\partial x_2}\right)+\sum z^{(k)}\left(c'\frac{\partial v^{(k)}}{\partial x_1}+d'\frac{\partial v^{(k)}}{\partial x_2}\right)=0,\end{aligned}$$

whence

$$(ab'-a'b)\sum y^{(k)}\frac{\partial u^{(k)}}{\partial x_2}+(ac'-a'c)\sum z^{(k)}\frac{\partial u^{(k)}}{\partial x_1}+(ad'-a'd)\sum z^{(k)}\frac{\partial u^{(k)}}{\partial x_2}=0,$$

$$(ab'-a'b)\sum y^{(k)}\frac{\partial v^{(k)}}{\partial x_1}+(cb'-c'b)\sum z^{(k)}\frac{\partial v^{(k)}}{\partial x_1}+(db'-d'b)\sum z^{(k)}\frac{\partial v^{(k)}}{\partial x_2}=0.$$

Remembering that we have assumed

$$\frac{\partial u^{(k)}}{\partial x_2}=\frac{\partial v^{(k)}}{\partial x_1},$$

we find by subtraction

$$\sum_{k=1}^{4} z^{(k)}\left[(ac'-a'c)\frac{\partial u^{(k)}}{\partial x_1}+(ad'-a'd)\frac{\partial u^{(k)}}{\partial x_2}+(bc'-b'c)\frac{\partial v^{(k)}}{\partial x_1}\right.$$
$$\left.+(bd'-b'd)\frac{\partial v^{(k)}}{\partial x_2}\right]=0.$$

By a similar process we can obtain another equation precisely the same except that $z^{(k)}$ is replaced by $y^{(k)}$. But any set of four quantities which satisfies two relations of the form

$$\sum y^{(k)}w^{(k)}=0,\quad \sum z^{(k)}w^{(k)}=0,$$

may be considered as the coordinates of a plane passing through L_{yz}. We may therefore write down the result that the quantities $u^{(k)}$, $v^{(k)}$ of our previous theorem satisfy a system of partial differential equations of the form

$$(ac'-a'c)\frac{\partial u}{\partial x_1}+(ad'-a'd)\frac{\partial u}{\partial x_2}+(bc'-b'c)\frac{\partial v}{\partial x_1}+(bd'-b'd)\frac{\partial v}{\partial x_2}$$
$$+\nu u+\rho v=0,\quad \frac{\partial u}{\partial x_2}-\frac{\partial v}{\partial x_1}=0\,. \tag{54}$$

The values of ν and ρ will depend upon the choice of the quantities $\lambda^{(1)}\ldots.\lambda^{(4)}$, $\mu^{(1)}\ldots.\mu^{(4)}$ subject to the conditions of the previous theorem.

It follows at once that the functions ϕ_k are solutions of the differential equation of the second order

$$k\frac{\partial^2\phi}{\partial x_1^2}+t\frac{\partial^2\phi}{\partial x_1\partial x_2}+l\frac{\partial^2\phi}{\partial x_2^2}+\nu\frac{\partial\phi}{\partial x_1}+\rho\frac{\partial\phi}{\partial x_2}=0\,, \tag{55}$$

if we remember the significance of the quantities k, l and t.

§8.—*Developable surfaces of the congruence.*

Suppose that we have four pairs of simultaneous solutions of system (1),

$$y^{(k)}=f^{(k)}(x_1,x_2),\quad z^{(k)}=g^{(k)}(x_1,x_2),\qquad (k=1,2,3,4) \tag{56}$$

the conditions being fulfilled under which they determine a congruence. If we put in (56),

$$x_1=\phi_1(t),\quad x_2=\phi_2(t), \tag{57}$$

we obtain two curves, c_y and c_z and those lines of the congruence which join corresponding points of the two curves are to be considered. They form a ruled surface of the congruence. The question arises as to the existence of developable surfaces made up of lines of the congruence.

If we imagine (57) substituted in (56), we find for every index k,

$$\frac{dy}{dt}=\frac{\partial y}{\partial x_1}\phi_1'+\frac{\partial y}{\partial x_2}\phi_2',\quad \frac{dz}{dt}=\frac{\partial z}{\partial x_1}\phi_1'+\frac{\partial z}{\partial x_2}\phi_2'.$$

But the surface in question is a developable surface if, and only if, the tangents of the two curves c_y and c_z constructed at corresponding points intersect, i. e. if, and only if, it is possible to satisfy the equation

$$\lambda y+\nu\frac{dy}{dt}+\mu z+\rho\frac{dz}{dt}=0\,,$$

where λ, μ, ν, ρ are functions of t, by putting $y=y^{(k)}$, $z=z^{(k)}$, for $(k=1, 2, 3, 4)$, i. e. if, and only if, λ, μ, ν, ρ can be determined so that

$$\lambda y+\mu z+\nu\left(\frac{\partial y}{\partial x_1}\phi_1'+\frac{\partial y}{\partial x_2}\phi_2'\right)+\rho\left(\frac{\partial z}{\partial x_1}\phi_1'+\frac{\partial z}{\partial x_2}\phi_2'\right)=0\,. \tag{58}$$

But y and z satisfy the equations

$$\Omega(y, z) = 0, \quad \Omega'(y, z) = 0.$$

We may combine these into an equation

$$\alpha\Omega(y, z) + \alpha'\Omega'(y, z) = 0$$

which becomes identical with (58) after substituting into it the values (57) for x_1 and x_2.

For the identity of these two equations we must have

$$\left.\begin{aligned} \alpha a + \alpha' a' &= \nu\phi_1', & \alpha c + \alpha' c' &= \rho\phi_1', & \alpha e + \alpha' e' &= \lambda, \\ \alpha b + \alpha' b' &= \nu\phi_2', & \alpha d + \alpha' d' &= \rho\phi_2', & \alpha f + \alpha' f' &= \mu. \end{aligned}\right\} \tag{59}$$

From the first four equations, eliminating α, α', ν and ρ, we obtain

$$\begin{vmatrix} a & a' & \phi_1' & 0 \\ b & b' & \phi_2' & 0 \\ c & c' & 0 & \phi_1' \\ d & d' & 0 & \phi_2' \end{vmatrix} = 0,$$

or, expanding and remembering the significance of k, l and t,

$$l\phi_1'^2 - t\phi_1'\phi_2' + k\phi_2'^2 = 0, \tag{60}$$

a quadratic differential equation, perhaps more clearly written

$$l\,dx_1^2 - t\,dx_1\,dx_2 + k\,dx_2^2 = 0. \tag{60a}$$

If this equation be satisfied by choosing x_2 as a function of x_1 in accordance with it, or more symmetrically x_1 and x_2 as functions of t, in accordance with (60), equations (59) can be satisfied. It is possible, therefore, to assemble the lines of the congruence into developable surfaces in an infinity of ways. *The congruence contains two families of* ∞^1 *developable surfaces, which are obtained by integrating the two differential equations obtained by factoring* (60).

Suppose that $k = l = 0$, $t \neq 0$. Then the two families of developables are obtained by putting either x_1 or x_2 equal to an arbitrary constant, i. e. the curves $x_1 =$ const. on the surfaces S_y and S_z are the curves in which the developables of one family intersect these surfaces, and the curves $x_2 =$ const. are the curves in which the developables of the other family intersect S_y and S_z. But equations (28) show that we may always choose new independent variables ξ_1 and ξ_2, so that for these $\bar{k}$ and $\bar{l}$ both vanish, if k and l are not zero in the first place, provided $t^2 - 4kl$ is not zero. We have merely to take for ξ_1 and ξ_2 two independent solutions of the equation

$$k\left(\frac{\partial\xi}{\partial x_1}\right)^2 + t\frac{\partial\xi}{\partial x_1}\,\frac{\partial\xi}{\partial x_2} + l\left(\frac{\partial\xi}{\partial x_2}\right)^2 = 0\,.$$

Therefore, if $t^2 - 4kl$ *does not vanish, and we introduce new independent variables* ξ_1 *and* ξ_2 *by putting*

$$\xi_1 = h_1(x_1, x_2), \quad \xi_2 = h_2(x_1, x_2),$$

where ξ_1 *and* ξ_2 *are two independent solutions of the differential equation*

$$k\left(\frac{\partial\xi}{\partial x_1}\right)^2 + t\frac{\partial\xi}{\partial x_1}\,\frac{\partial\xi}{\partial x_2} + l\left(\frac{\partial\xi}{\partial x_2}\right)^2 = 0\,, \tag{61}$$

the curves $\xi_1 =$ *const. and* $\xi_2 =$ *const. represent the curves on both surfaces* S_y *and* S_z *in which these surfaces intersect the two families of developable surfaces of the congruence. Moreover this choice of parameters is characterized by the conditions* $k = l = 0$.

It must be possible to determine the functions $\alpha, \beta, \gamma, \delta$ in the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta \tag{62}$$

in such a way that the surfaces S_η and S_ζ may become those two surfaces which contain the edges of regression of the two families of developable surfaces. Suppose that the surface S_η is the locus of the edges of regression of that family of developables corresponding to $\xi_1 =$ const. and S_ζ that of the developables corresponding to $\xi_2 =$ const. Then P_ζ must lie on a tangent to the curve $\xi_1 =$ const. which passes through P_η, i. e. an equation of the form

$$\bar{b}\frac{\partial\eta}{\partial\xi_2} + \bar{e}\eta + \bar{f}\zeta = 0, \tag{63a}$$

must be satisfied, and similarly an equation of the form

$$\bar{c}' \frac{\partial \zeta}{\partial \xi_1} + \bar{e}' \eta + \bar{f}' \zeta = 0. \tag{63b}$$

Moreover, we have already assumed that the preliminary reduction

$$k = l = 0$$

has been made. We must therefore have

$$\bar{a} = 0, \quad \bar{c} = \bar{d} = 0, \quad \bar{a}' = \bar{b}' = 0, \quad \bar{d}' = 0, \quad \bar{k} = \bar{l} = 0,$$

where $\bar{a}$, etc., are expressed in terms of a, b, c, etc., by equations (6).

The conditions $\bar{c} = \bar{d} = 0$ and $\bar{a}' = \bar{b}' = 0$ give $i = i' = 0$ respectively, for neither β and δ nor α and γ can vanish simultaneously, as that would make $\alpha\delta - \beta\gamma$ vanish. We can therefore put

$$\beta = -d, \quad \delta = b, \quad \alpha = -c', \quad \gamma = a',$$

and then find

$$\bar{a} = -(ac' - a'c) = -k = 0, \quad \bar{d}' = bd' - b'd = l = 0,$$

so that all of the conditions are satisfied.

The lines of the congruences are clearly the common tangents of the two surfaces S_η and S_ζ thus determined. These are usually looked upon as two sheets of a single surface, the *focal surface* of the congruence.

If, then, for a system of form (1) the conditions $k = l = i = i' = 0$ are satisfied, the two sheets of the focal surface of the congruence may be obtained by making the transformation

$$y = c'\eta + d\zeta, \quad z = a'\eta + b\zeta.$$

If t also vanishes, they coincide. The curves $x_1 =$ const. on S_η and the curves $x_2 =$ const. on S_ζ are the cuspidal edges of the two families of developables which are contained in the congruence.

We can, however, obtain the focal surface in another, more elegant manner.

Let the system (1) or

$$\Omega = 0, \quad \Omega' = 0$$

be given in the first place, and let its coefficients be a, b, c, etc. Consider in place of this the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0,$$

whose coefficients A, B, C, etc., are given by (21). Then transform this system by putting

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

and denote the new coefficients by $\overline{A}$, $\overline{B}$, $\overline{C}$, etc. We shall then have, in particular

$$\left.\begin{aligned} \overline{C} &= (a\beta + c\delta)\phi + (a'\beta + c'\delta)\psi, & \overline{A}' &= (a\alpha + c\gamma)\chi + (a'\alpha + c'\gamma)\omega, \\ \overline{D} &= (b\beta + d\delta)\phi + (b'\beta + d'\delta)\psi, & \overline{B}' &= (b\alpha + d\gamma)\chi + (b'\alpha + d'\gamma)\omega. \end{aligned}\right\} \quad (64)$$

Let us determine these transformations, so that

$$\overline{C} = \overline{D} = \overline{A}' = \overline{B}' = 0. \quad (65)$$

Then the new system has the form

$$\left.\begin{aligned} \overline{A}\frac{\partial\eta}{\partial x_1} + \overline{B}\frac{\partial\eta}{\partial x_2} + \overline{E}\eta + \overline{F}\zeta = 0, \\ \overline{C}'\frac{\partial\zeta}{\partial x_1} + \overline{D}'\frac{\partial\zeta}{\partial x_2} + \overline{E}'\eta + \overline{F}'\zeta = 0. \end{aligned}\right\} \quad (66)$$

The first equation shows that P_ζ is in the plane tangent to S_η at P_η, and the second shows that P_η is in the plane tangent to S_ζ at P_ζ. In other words, the line $P_\eta P_\zeta$ is tangent to both S_η and S_ζ, which must therefore be the two sheets of the focal surface.

But in order that equations (65) may be satisfied it is necessary and sufficient, (since the possibilities $\phi = \psi = 0$, and $\chi = \omega = 0$ are excluded, and since the determinant $\alpha\delta - \beta\gamma$ must not vanish) that the ratios $\frac{\beta}{\delta}$ and $\frac{\alpha}{\gamma}$ should be distinct roots of the quadratic

$$(ab' - a'b)\lambda^2 + (ad' - bc' + cb' - da')\lambda + cd' - c'd = 0,$$

whose discriminant is $t^2 - 4kl$.

But we have

$$(\alpha\delta-\beta\gamma)\eta=\delta y-\beta z,\quad (\alpha\delta-\beta\gamma)\zeta=-\gamma y+\delta z,$$

whence

$$(\alpha\delta-\beta\gamma)^2\eta\zeta=\gamma\delta\left[-y^2-\frac{\alpha}{\gamma}\,\frac{\beta}{\delta}z^2+\left(\frac{\alpha}{\gamma}+\frac{\beta}{\delta}\right)yz\right]$$

$$=\frac{-\gamma\delta}{ab'-a'b}\left[(ab'-a'b)\,y^2+(cd'-c'd)\,z^2+(ad'-bc'+cb'-da')\,yz\right].$$

We have then the following result. *The factors of the expression*

$$C=(ab'-a'b)\,y^2+(ad'-a'd+cb'-c'b)\,yz+(cd'-c'd)\,z^2 \tag{67}$$

are distinct if $t^2-4kl\neq 0$. *If they are put equal to* η *and* ζ *respectively, the surfaces* S_η *and* S_ζ *are the two sheets of the focal surface of the congruence.*

One may conclude herefrom that the above expression is a *covariant* of our system. This may be verified analytically.

Moreover we learn that the condition

$$t^2-4kl=0$$

means that the two sheets of the focal surface of the congruence coincide.

If this reduction be combined with that of our previous theorem, we notice that every system of form (1) can be reduced to the form

$$\left.\begin{aligned} y_2&=\lambda y+\mu z,\\ z_1&=\nu y+\rho z,\end{aligned}\right\} \tag{68}$$

provided that $t^2-4kl\neq 0$. In that case the surfaces S_y and S_z are the two (distinct) sheets of the focal surface. Moreover, the curves $x_1=$ const. on S_y and the curves $x_2=$ const. on S_z are the cuspidal edges of the two families of developable surfaces of the congruence.

The curves $x_1=$ const. and $x_2=$ const. are conjugate curves on each of these surfaces. For, if we differentiate both members of the first equation of

(68) with respect to x_1 and those of the second with respect to x_2, we find

$$\left.\begin{aligned} \mu y_{12} &= \lambda\mu y_1 + (\mu_1 + \mu\rho)\, y_2 + [\mu(\lambda_1 + \mu\nu) - \lambda(\mu_1 + \mu\rho)]\, y, \\ \nu z_{12} &= (\nu_2 + \lambda\nu)\, z_1 + \nu\rho z_2 + [\nu(\rho_2 + \mu\nu) - \rho(\nu_2 + \lambda\nu)]\, z. \end{aligned}\right\} \qquad (69)$$

But any four functions of x_1, x_2 which satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial x_1 \partial x_2} + a\frac{\partial\theta}{\partial x_1} + b\frac{\partial\theta}{\partial x_2} + c\theta = 0$$

can be taken as the four homogeneous coordinates of the points of a surface on which the curves $x_1 =$ const. and $x_2 =$ const. are conjugate lines.*

If the sheet of the focal surface S_η degenerates into a curve, it must be possible to find a function of x_1 and x_2, such that the ratios of $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$, $\eta^{(4)}$ are functions of θ alone. This is the case if in equations (66), $\bar{F} = 0$. For let $\eta^{(i)}$, $\eta^{(k)}$ be any two solutions of the equation

$$\bar{A}\frac{\partial\eta}{\partial x_1} + \bar{B}\frac{\partial\eta}{\partial x_2} + \bar{E}\eta = 0,$$

then their quotient satisfies the equation

$$\bar{A}\frac{\partial}{\partial x_1}\frac{\eta^{(i)}}{\eta^{(k)}} + \bar{B}\frac{\partial}{\partial x_2}\frac{\eta^{(i)}}{\eta^{(k)}} = 0,$$

whose most general solution is an arbitrary function of any particular solution θ. Moreover any four functions whose ratios are functions of θ alone always satisfy an equation of this form, so that the condition $\bar{F} = 0$ is necessary as well as sufficient.

To express this condition in invariant form, let us assume in the first place that $i = 0$. We have seen that we can always assume that this is the case by choosing the functions ϕ, ψ, χ and ω properly and considering the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0,$$

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* Darboux, Théorie des Surfaces, Tome I, p. 122.

instead of $\Omega = \Omega' = 0$. Then make the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta.$$

The first equation should assume the form

$$\bar{a}\frac{\partial\eta}{\partial x_1} + \bar{b}\frac{\partial\eta}{\partial x_2} + \bar{e}\eta = 0.$$

Therefore, we must have $\bar{c} = \bar{d} = \bar{f} = 0$, or

$$\left.\begin{aligned} &a\beta + c\delta = 0, \quad b\beta + d\delta = 0, \\ &a\beta_1 + b\beta_2 + c\delta_1 + d\delta_2 + e\beta + f\delta = 0. \end{aligned}\right\} \tag{70}$$

The first two conditions can be fulfilled by putting

$$\beta : \delta = -c : a = -d : b,$$

since

$$i = ad - bc = 0.$$

If we differentiate each of these conditions with respect to x_1 and x_2 and use the resulting equations to eliminate β_1, β_2, δ_1, δ_2 from the third condition, we find

$$(e - a_1 - b_2)\beta + (f - c_1 - d_2)\delta = 0,$$

or, according to (12),

$$p\beta + q\delta = 0, \tag{71}$$

which, together with the first two conditions, gives

$$r = aq - cp = 0, \quad s = bq - dp = 0, \tag{72}$$

for β and δ cannot vanish simultaneously. On the other hand, if these conditions are satisfied, one finds $\bar{c} = \bar{d} = \bar{f} = 0$ on making the above transformation.

Therefore, the conditions $r = 0$, $s = 0$, are necessary and sufficient to make one of the sheets of the focal surface degenerate if $i = 0$.

If we refer everything to the normal form for which $i = i' = 0$, which always exists if $t^2 - 4kl \neq 0$, we see that *if* $r = 0$, $s = 0$, *the sheet* S_η *of the focal surface degenerates, if* $r' = 0$, $s' = 0$ *the sheet* S_ξ *degenerates. If* $r = r' = s = s' = 0$ *both sheets of the surface become curves;* if $t^2 - 4kl$ also vanishes, these two focal curves coincide.

If the system of differential equations be reduced to its canonical form (68) for $t^2 - 4kl \neq 0$, if $r = 0$, μ must vanish, and the first equation becomes integrable by quadratures. Similarly the second for $r' = 0$.

Therefore one of the equations of our system becomes integrable by quadratures if, when written in the canonical form, either r and s or r' and s' vanish, i. e. if one of the sheets of the focal surface reduces to a curve. Both equations are integrable by quadratures if r, s and r', s' both vanish, i. e. if both sheets of the focal surface reduce to curves. This presupposes however a reduction to the canonical form, which requires the factoring of the quadratic covariant C (equation (67)), and the solution of equation (61), which is equivalent to the solution of the ordinary differential equation (60a).

Let us construct a canonical form for the case $t^2 - 4kl = 0$. We can assume for the first equation the same canonical form as before, so that our system will certainly have the form

$$by_2 + ey + fz = 0, \quad b \neq 0,$$
$$a'y_1 + b'y_2 + c'z_1 + d'z_2 + e'y + f'z = 0,$$

whence $\quad k = 0, \quad l = bd', \quad m = 0, \quad n = -bc', \quad t = bc'.$

But in this case $t^2 - 4kl = 0$, so that c' must vanish. But then we can eliminate y_2 from the second equation by means of the first, which gives the canonical form sought

$$\left.\begin{aligned} y_2 &= \lambda y + \mu z, \\ a'y_1 + d'z_2 &= \nu y + \rho z. \end{aligned}\right\} \tag{73}$$

By differentiation we find

$$\mu d' y_{22} = \mu^2 a' y_1 + [d'(\mu_2 + \lambda\mu) - \rho\mu] y_2 + [(\lambda_2\mu - \mu_2\lambda)d' + \mu(\lambda\rho - \mu\nu)] y, \tag{74}$$

which proves that *the lines of a congruence, the sheets of whose focal surface coincide,* *are tangents to one of the two sets of asymptotic lines of that surface.* For, any four solutions of an equation of the form

$$\theta_{22} = a\theta_1 + b\theta_2 + c\theta$$

define a surface upon which the curves $x_1 = \text{const.}$ are asymptotic lines.*

An exceptionally interesting example can be given for such systems $\Omega = 0$, $\Omega' = 0$, for which both sheets of the focal surface degenerate. For, consider the system, well known in the theory of functions,

$$\Omega = \frac{\partial y}{\partial x_2} + \frac{\partial z}{\partial x_1} = 0, \quad \Omega' = \frac{\partial y}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0, \tag{75}$$

for which we find

$$i = -1, \quad i' = -1, \quad k = -1, \quad l = -1, \quad m = n = t = 0, \quad t^2 - 4kl = -4$$

In order to make i and i' vanish, we consider the system

$$\phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0,$$

instead, where we put

$$\phi = \sqrt{-1}, \quad \psi = 1,$$
$$\chi = -\sqrt{-1}, \quad \omega = 1,$$

which gives us

$$\frac{\partial y}{\partial x_1} + \sqrt{-1}\,\frac{\partial y}{\partial x_2} + \sqrt{-1}\,\frac{\partial z}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0,$$
$$\frac{\partial y}{\partial x_1} - \sqrt{-1}\,\frac{\partial y}{\partial x_2} - \sqrt{-1}\,\frac{\partial z}{\partial x_1} - \frac{\partial z}{\partial x_2} = 0,$$

for which

$$k = -2\sqrt{-1}, \quad l = -2\sqrt{-1}, \quad m = -2, \quad n = -2, \quad t = 0, \quad t^2 - 4kl = 16,$$
$$i = i' = r = s = r' = s' = 0,$$

so that its focal surface degenerates into two distinct curves.

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* Darboux. Théorie des Surfaces, Tome I, p. 144.

To find these curves we make the substitution obtained by factoring the covariant C, i. e. we put

$$y=\sqrt{-1}\eta+\sqrt{-1}\zeta, \quad z=\eta-\zeta,$$

which gives

$$\sqrt{-1}\frac{\partial\eta}{\partial x_1}-\frac{\partial\eta}{\partial x_2}=0, \quad \sqrt{-1}\frac{\partial\zeta}{\partial x_1}+\frac{\partial\zeta}{\partial x_2}=0, \tag{76}$$

and therefore

$$\eta^{(k)}=f^{(k)}(x_1+ix_2), \quad \zeta^{(k)}=g^{(k)}(x_1-ix_2), \qquad (k=1,\ 2,\ 3,\ 4), \tag{77}$$

where the $f^{(k)}$ and $g^{(k)}$ functions are arbitrary functions of the two independent arguments x_1+ix_2 and x_1-ix_2, so that equations (77) represent two arbitrary curves, whose points are put into an arbitrary correspondence with each other.

Therefore, the partial differential equations of the theory of functions represent any congruence whatever, which is made up of the lines intersecting two distinct curves. Any system of partial differential equations of form (1), *for which* $i=i'=r=r'=0$, *can be transformed into the canonical form*

$$\frac{\partial y}{\partial x_2}+\frac{\partial z}{\partial x_1}=0, \quad \frac{\partial y}{\partial x_1}-\frac{\partial z}{\partial x_2}=0.$$

We can make another interesting application. Let us suppose that $t^2-4kl\neq 0$, and that the sheet S_y of the focal surface is a surface of translation, which may be generated either by the curves $x_1=\text{const.}$ or $x_2=\text{const.}$ Moreover let us assume that our coordinates are cartesian. Then the first equation of (69) must reduce to $y_{12}=0$. We must, therefore, have

$$\lambda=0, \quad \mu_1+\mu\rho=0, \quad \lambda_1+\mu\nu=0,$$

for μ cannot vanish, since the surface S_y would otherwise degenerate into a curve, as shown by (68). Therefore, we find $\nu=0$, i. e. the second sheet of the focal surface is a curve. We can therefore enunciate the following theorem:

Consider the congruences formed by the tangents of the curves $x_1=$ *const. and* $x_2=$ *const. of a surface of translation, the surface being capable of being generated* *by the translation of either a curve $x_1 = const.$ or $x_2 = const.$ All of the lines of such a congruence intersect a curve.*

This theorem is essentially the converse of a theorem of Lie's.*

§9.—*Catalogue of types.*

We have seen that, if $t^2 - 4kl \neq 0$, system (1) can be reduced to the form

$$y_2 = \lambda y + \mu z, \quad z_1 = \nu y + \rho z.$$

But this can be simplified. For, if we put

$$y = \bar{y} \, . \, e^{\int \lambda dx_2}, \quad z = \bar{z} e^{\int \rho dx_1},$$

we find for $\bar{y}$ and $\bar{z}$ a system of the form

$$\bar{y}_2 = \bar{\mu}\bar{z}, \quad \bar{z}_1 = \bar{\nu}\bar{y}. \tag{78}$$

If $t^2 - 4kl = 0$, we have already found for the system the form

$$\left.\begin{aligned} y_2 &= \lambda y + \mu z, \\ a' y_1 + d' z_2 &= \nu y + \rho z. \end{aligned}\right\} \tag{73}$$

Let us assume $d' \neq 0$, and put

$$y = \bar{y} e^{\int \lambda dx_2}, \quad z = \bar{z} e^{\int -\frac{\rho}{d'} dx_2}.$$

Then (73) becomes

$$\bar{y}_2 = \bar{\mu}\bar{z}, \quad \bar{a}\bar{y}_1 + \bar{z}_2 = \bar{\nu}\bar{y}. \tag{79}$$

If $d' = 0$, (73) becomes

$$y_2 = \lambda y + \mu z, \quad a' y_1 = \nu y + \rho z.$$

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* Lie-Scheffers, Geometrie der Berührungstransformationen, p. 383.

We may now assume $a' \neq 0$, for otherwise we could at once reduce our system of equations to a single equation of the first order. We may therefore put

$$y = \bar{y} e^{\int \frac{\nu}{a'} dx_1}, \quad z = \bar{z},$$

which gives

$$\bar{y}_2 = \bar{\lambda} \bar{y} + \bar{\mu} \bar{z}, \quad \bar{y}_1 = \bar{\rho} \bar{z}. \tag{80}$$

In this case the congruence degenerates. For we have

$$-\bar{\mu} \bar{y}_1 + \bar{\rho}\, \bar{y}_2 = \bar{\rho} \bar{\lambda} \bar{y}, \quad \bar{\rho} \bar{z} = \bar{y}_1 .$$

The first equation shows that the focal surface is, in this case, a curve. The second shows that P_z is always on the tangent to this curve constructed at P_y. Instead of ∞^2 straight lines, we have therefore merely ∞^1, the tangents of the curve described by P_y. By writing $\bar{\rho} \bar{z} = z'$ and then dropping the strokes, we can write for this type more simply

$$y_1 = z, \quad y_2 = \lambda y + \mu z. \tag{81}$$

We have found, therefore, the following typical forms for systems of partial differential equations of form (1):

$$\left.\begin{array}{rlll} \text{I.} & t^2 - 4kl \neq 0, & y_2 = \mu z, & z_1 = \nu y; \\ \text{II.} & t^2 - 4kl = 0, & y_2 = \mu z, & a y_1 + z_2 = \nu y; \\ \text{III.} & t^2 - 4kl = 0, & y_1 = z\ , & y_2 = \lambda y + \mu z. \end{array}\right\} \tag{82}$$

Any system of form (1), which cannot be reduced to a single equation, can be reduced to one of these forms. It now becomes a simple matter to decide whether two such systems are equivalent. The systems are reduced to their typical forms and these are compared. If the types are equivalent, the same thing is true of the original systems.

§10.—*Laplace transformations.*

A linear differential equation of the form

$$\frac{\partial^2 z}{\partial x\, \partial y} + a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + c z = 0, \tag{83}$$

can be transformed into another of the same form by either of the transformations

$$z_1 = \frac{\partial z}{\partial y} + az, \quad z_{-1} = \frac{\partial z}{\partial x} + bz.^* \tag{84}$$

Thus from any equation of this form a single infinity of other equations is obtained by the transformation of Laplace. The original equation has two invariants

$$h = \frac{\partial a}{\partial x} + ab - c, \quad k = \frac{\partial b}{\partial y} + ab - c,$$

and if either of these is zero, the equation can be integrated by quadratures. Each of the equations, obtained from the given one by the method of Laplace, of course, also has two invariants, and if either of these vanishes, that equation and, therefore, the original one can be integrated by quadratures. A great many interesting investigations can be attached to the theory of these transformations.

M. Darboux has interpreted the transformation geometrically, in a beautiful manner, and it is on the basis of this geometrical interpretation that we shall introduce Laplace transformations of our system.

If $z^{(1)} \ldots . z^{(4)}$ are 4 solutions of (83) and are taken as the homogeneous coordinates of a point in space, then the locus of this point is a surface upon which the curves $x =$ const., $y =$ const., form a conjugate system. Consider the curves $x =$ const. and construct their tangents. They form a congruence one sheet of whose focal surface is the given surface. Clearly the tangents along a curve $x =$ const. form a developable surface of the congruence whose cuspidal edge is that curve $x =$ const. But we know that the lines of the congruence can be assembled into developable surfaces in another way. Consider any fixed curve $y =$ const. It intersects the curves $x =$ const. At each of the points of intersection, construct the tangent to the curve $x =$ const. at that point. These lines of course belong to the congruence, and form a developable surface whose cuspidal edge lies on the other sheet of the focal surface. Now it is this other sheet which is obtained by putting

$$z_1 = \frac{\partial z}{\partial y} + az.$$

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* Darboux, Théorie des surfaces. T. II, Chap. 1.

In the same way the second transformation arises from the congruence of lines tangent to the curves $y =$ const.

Our transformations are precisely the same as these, geometrically. We assume that $t^2 - 4kl \neq 0$, so that we may write for our system

$$\frac{\partial y}{\partial x_2} = \mu z, \quad \frac{\partial z}{\partial x_1} = \nu y. \tag{85}$$

The surfaces S_y and S_z are the distinct sheets of the focal surface of our congruence C. The curves $x_1 =$ const. on S_y and the curves $x_2 =$ const. on S_z are the cuspidal edges of the two families of developable surfaces. We wish to deduce from the given congruence C another one C_1 which shall have the surface S_z in common with C as one sheet of its focal surface. But we shall assume that in this second congruence the curves $x_1 =$ const. on S_z instead of $x_2 =$ const. shall be the cuspidal edges of the one set of developable surfaces. Every line of the new congruence is then tangent to a curve $x_1 =$ const. on the surface S_z and, therefore joins P_z to a point on such a tangent, which we shall call P_ζ. We must therefore put

$$\eta = z, \quad \zeta = \alpha z + \frac{\partial z}{\partial x_2}. \tag{86}$$

But as P_z moves along a curve $x_2 =$ const., the line $P_z P_\zeta$ must describe a developable surface of the congruence C, which has its edge of regression on the, as yet unknown, second sheet of its focal surface. We wish to determine λ so that P_ζ shall be a point of this edge of regression. This is so if, and only if, during this motion the line $P_z P_\zeta$ always remains tangent to the curve described by P_ζ, i. e. if an equation of the form

$$\frac{\partial \zeta}{\partial x_1} = \beta z + \gamma \zeta \tag{87}$$

be fulfilled. If we substitute for ζ the value from (86), we find

$$z_{12} + \alpha z_1 + \alpha_1 z = \beta z + \gamma \alpha z + \gamma z_2,$$

or

$$z_{12} + \alpha z_1 - \gamma z_2 + (\alpha_1 - \beta - \gamma \alpha) z = 0.$$

But, on the other hand, we have, if we put $\lambda = \rho = 0$ in (69),

$$\nu z_{12} = \nu_2 z_1 + \mu \nu^2 z$$

or

$$z_{12} - \frac{\nu_2}{\nu} z_1 - \mu \nu z = 0.$$

The two equations must be identical, as they represent the same surface S_z referred to the same conjugate system. Therefore,

$$\alpha = -\frac{\nu_2}{\nu}, \quad \gamma = 0, \quad \alpha_1 - \beta = -\mu\nu,$$

whence we see that the transformation (86) becomes

$$\eta = z, \quad \zeta = -\frac{\nu_2}{\nu} z + \frac{\partial z}{\partial x_2}. \tag{88}$$

Equations (87) and (86) give

$$\frac{\partial \zeta}{\partial x_1} = \left[\mu\nu - \frac{\partial^2 \log \nu}{\partial x_1 \partial x_2}\right]\eta, \quad \frac{\partial \eta}{\partial x_2} = \zeta + \frac{\partial \log \nu}{\partial x_2}\eta.$$

If we introduce $\frac{1}{\nu}\eta$ in place of η, we obtain the following result. The congruence C_1 is obtained by making the transformation

$$\eta = \frac{z}{\nu}, \quad \zeta = -\frac{\nu_2}{\nu} z + \frac{\partial z}{\partial x_2}, \tag{89}$$

and its equations are

$$\frac{\partial \eta}{\partial x_2} = \frac{1}{\nu}\zeta, \quad \frac{\partial \zeta}{\partial x_1} = \left(\mu\nu - \frac{\partial^2 \log \nu}{\partial x_1 \partial x_2}\right)\nu\eta. \tag{89a}$$

Similarly another congruence C_{-1} is obtained by putting

$$\eta = -\frac{\mu_1}{\mu} y + \frac{\partial y}{\partial x_1}, \quad \zeta = \frac{y}{\mu}, \tag{90}$$

viz.

$$\frac{\partial \eta}{\partial x_2} = \left(\mu\nu - \frac{\partial^2 \log u}{\partial x_1 \partial x_2}\right)\mu\zeta, \quad \frac{\partial \zeta}{\partial x_1} = \frac{1}{\mu}\eta. \tag{90a}$$

The repetition of the first transformation gives rise to a sequence of congruences C_1, C_2, C_3, ... and the second gives rise to a similar sequence C_{-1}, C_{-2}, C_{-3}, ... etc. If this sequence is finite in either direction, one of the sheets of the focal surface of the last congruence thus obtained degenerates into a curve. This takes place for C_1 for instance if

$$\mu\nu - \frac{\partial^2 \log \nu}{\partial x_1 \partial x_2} = 0.$$

It should be noted that for our system, the quantities r, s, r', s', become

$$r=0, \quad s=-\mu, \quad r'=+\nu, \quad s'=0,$$

so that clearly the coefficients of the various systems obtained by Laplace transformations are expressible in terms of r' and s, which take the place of the quantities h and k in the theory of the differential equation of Laplace.

The entire theory of the differential equation of Laplace can clearly be presented from this new point of view.

Moreover it should be noted that, although we have used the canonical form of our system of differential equations to deduce the form of the Laplace transformations, we might have used its unreduced form. Then it is clear that the various dependent variables introduced by the successive Laplace transformations will be connected covariantly with our system of partial differential equations.

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